Int. J. Solids Structures, 1973, Vol. 9, pp. 527 to 551. Pergamon Press. Printed in Great Britain

BENDING AND TORSION OF ANISOTROPIC BEAMS

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Abstract—A mathematical formulation of the bending and torsion of an anisotropic elastic beam is given. Approximate analytic solutions to the problem are obtained by variational methods and are shown to agree well with a numerical solution. The results are used to analyse the bending and torsion of a transversely isotropic elastic beam. It is shown that the bending stiffness of a beam is greater when twisting is prevented—a result which is significant for design with composite materials. A sequence of bending and torsion experiments is described from which the five compliances of a transversely isotropic material may be determined. The results are generalized in order to determine the frequency dependent compliances of anisotropic viscoelastic materials.

1. INTRODUCTION

IN RECENT years there has been a considerable revival of interest in anisotropic elasticity theory. This is a result of the need to analyse the mechanical behaviour of laminated and fibre reinforced materials. In many applications these materials behave as anisotropic elastic solids whose mechanical response is governed by a number of independent elastic constants. Techniques are available for determining these constants by ultrasonic pulse techniques (see, for example, [1]), but there is a need for elementary static tests which use standard laboratory equipment. For isotropic materials, bending and torsion tests determine the elastic constants with a minimum of specimen preparation. In this paper a mathematical basis for bending and torsion tests of anisotropic elastic beams with monoclinic symmetry is provided by analysing the mechanical behaviour of such beams when subjected to bending and twisting moments.

Three conclusions of practical importance emerge from the analysis. The main interest is centred on a method for determining the five elastic constants of a transversely isotropic material by a sequence of bending and torsion tests. In addition, by noting a correspondence between the governing elastic equations and those of anisotropic viscoelasticity theory we are able to extend our results to the analysis of time harmonic bending and torsion tests for viscoelastic beams. This provides one of the first practical methods for determining all of the frequency dependent compliances for a transversely isotropic material. Since many composite materials are both anisotropic and viscoelastic this extension of our analysis is important in the testing of composite materials. Finally, by exploiting the coupling which exists between bending and torsion, we show how a beam may have a bending stiffness which is greater than the maximum modulus associated with the material of the beam.

In Section 2 we set out the basic equations and boundary conditions for the equilibrium response of a monoclinic elastic beam subjected to both bending and twisting moments. We show in Section 3 how these equations apply to a transversely isotropic elastic beam in which the symmetry axis of the material is inclined to the axis of the beam. It is not possible to solve the resulting equations analytically. We, therefore, present in Section 4 a variational formulation of the problem, which is more useful when obtaining approximate solutions. The variational problem is solved in Section 5 and approximate expressions are derived which relate the deflection and twist of the beam to the applied bending and twisting moments. The accuracy of these formulae is tested by comparison with a computed solution to the full equations. The main results are summarized in Section 6 where a sequence of experiments is described which systematically determines the five elastic constants of a transversely isotropic material, and the extension to anisotropic viscoelastic materials is noted.

2. FORMULATION OF THE PROBLEM

(a) *Basic equations*

We consider a homogeneous, anisotropic, elastic beam of length l and of rectangular cross-section with sides a, b. The beam is supported at each end and is in equilibrium under the influence of a surface stress distribution. We refer the beam to a system of rectangular Cartesian coordinates whose origin 0 is at the centre of one end with the x_3-axis parallel to the long sides of the beam, (see Fig. 1). We assume that there are no body forces, that stresses acting on the ends of the beam are equipollent to a constant bending moment M about the x_1-axis and a constant twisting moment M_t about the x_3-axis† and that the long sides of the beam are stress-free.

The elastic material of the beam is assumed to possess monoclinic symmetry (13 independent elastic constants), with $x_2 = 0$ as the single plane of symmetry. We note, however, that the methods described in this paper also apply when the material has a general anisotropy.

Our aim is to determine the deformed position of the beam under the influence of the bending moment M and twisting moment M_t.

Let t_{ij} and e_{ij} denote the stress and strain tensors in the beam, where the subscripts take the values 1, 2, 3 and the summation convention applies to repeated indices. We assume that the beam undergoes only small deformations from a stress-free reference state, so that the linearized equations of classical elasticity theory may be applied. We must, therefore, solve the equilibrium equations

$$t_{ij,j} = 0, \tag{2.1}$$

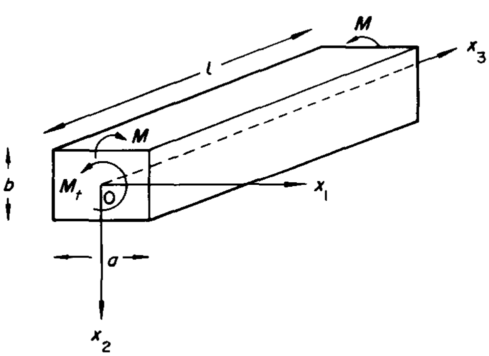

FIG. 1. Configuration of the beam. (Anticlockwise moments are taken to be positive).

† We invoke St. Venant's principle and neglect any effects associated with the clamping of the ends of the beam. This assumption is valid for long beams.

and the strain compatibility equations

$$e_{ij,kl}+e_{kl,ij}-e_{ik,jl}-e_{jl,ik} = 0, \tag{2.2}$$

with the constitutive relations

$$e_{ij} = a_{ijkl}t_{kl}. \tag{2.3}$$

The $,i$ notation used here denotes $\partial/\partial x_i$. The elastic compliance tensor a_{ijkl} satisfies the usual symmetry relations $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$, hence for a monoclinic material it can be shown that the 81 components of a_{ijkl} reduce to 13 independent constants. The strain tensor is related to the displacement vector u_i by

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{2.4}$$

Necessary and sufficient conditions for the integration of these equations for the displacements are the compatibility conditions (2.2).

Since the beam is in equilibrium and the applied forces are independent of x_3, each cross-section is subjected to the same constant moments M, M_t and, hence the stresses t_{ij} in the beam depend only on the coordinates x_1, x_2.† It follows from the constitutive equations that the strains e_{ij} are also functions of x_1 and x_2. The beam is said to be in a state of generalized plane stress. A detailed account of an anisotropic elastic cylinder in generalized plane stress has been given by Lekhnitskii [2] where some aspects of the following analysis are discussed. On making the assumptions $t_{ij} = t_{ij}(x_1, x_2)$, $e_{ij} = e_{ij}(x_1, x_2)$, the equilibrium equations (2.1) reduce to

$$\begin{aligned} t_{11,1}+t_{12,2} &= 0, \\ t_{12,1}+t_{22,2} &= 0, \\ t_{13,1}+t_{23,2} &= 0, \end{aligned} \tag{2.5}$$

the compatibility conditions (2.2) become

$$\begin{aligned} e_{11,22}+e_{22,11} &= 2e_{12,12}, \\ e_{33,11} = e_{33,22} &= e_{33,12} = 0, \\ (e_{13,2}-e_{23,1})_{,1} &= (e_{13,2}-e_{23,1})_{,2} = 0, \end{aligned} \tag{2.6}$$

and the constitutive equations (2.3) may be written out in full as

$$\begin{aligned} e_{11} &= a_{11}t_{11}+a_{12}t_{22}+a_{13}t_{33}+a_{15}t_{13}, \\ e_{22} &= a_{12}t_{11}+a_{22}t_{22}+a_{23}t_{33}+a_{25}t_{13}, \\ e_{33} &= a_{13}t_{11}+a_{23}t_{22}+a_{33}t_{33}+a_{35}t_{13}, \\ 2e_{23} &= a_{44}t_{23}+a_{46}t_{12} \\ 2e_{13} &= a_{15}t_{11}+a_{25}t_{22}+a_{35}t_{33}+a_{55}t_{13}, \\ 2e_{12} &= a_{46}t_{23}+a_{66}t_{12}. \end{aligned} \tag{2.7}$$

† A constant bending moment M is realized in practice by subjecting the beam to a four point bending test. If the beam were a cantilever or in three point bending, then $M = M(x_3)$ and the analysis presented here would require modification.

where $a_{\alpha\beta}$ ($\alpha, \beta = 1 \ldots, 6$) are the usual contracted notation for the a_{ijkl}, see Hearmon ([3], Section 1.3). We note that for a monoclinic elastic material, with the plane of symmetry $x_2 = 0$, there are 13 independent non-zero compliances. These may be obtained from the 36 $a_{\alpha\beta}$ on using the relation $a_{\alpha\beta} = a_{\beta\alpha}$ and on setting

$$a_{14} = a_{16} = a_{24} = a_{26} = a_{34} = a_{36} = a_{45} = a_{56} = 0.$$

Equations $(2.6)_{2,3}$ may be integrated immediately to give

$$e_{33} = a_{33}(Ax_1 + Bx_2 + C),$$
$$e_{13,2} - e_{23,1} = -\alpha, \tag{2.8}$$

where A, B, C and α are constants of integration, whose physical meaning will become apparent later. Equations (2.5), $(2.6)_1$, (2.7) and (2.8) now provide 12 scalar equations for the determination of the 6 components of stress and 6 components of strain in the beam.

These 12 equations are now reduced to a pair of equations in the scalar stress function $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$. For, on introducing the Airy stress function ϕ and a shear stress function ψ in terms of which the stress components are

$$t_{11} = \phi_{,22}, \quad t_{22} = \phi_{,11}, \quad t_{12} = -\phi_{,12},$$
$$t_{13} = \psi_{,2}, \quad t_{23} = -\psi_{,1}, \tag{2.9}$$

it follows that the equilibrium equations (2.5) are satisfied identically. The axial stress t_{33} becomes

$$t_{33} = Ax_1 + Bx_2 + C - \frac{1}{a_{33}}(a_{13}\phi_{,22} + a_{23}\phi_{,11} + a_{35}\psi_{,2}), \tag{2.10}$$

when $(2.8)_1$ and (2.9) are substituted into equation $(2.7)_3$. All the components of stress and strain are now defined in terms of ϕ and ψ. Any arbitrary functions ϕ and ψ determine stresses which satisfy the equilibrium equations. However, these stresses are only admissible in an elastic body when the two remaining compatibility conditions for the strains are satisfied. On substituting for the e_{ij} in equations $(2.6)_1$ and $(2.8)_2$ we obtain the following coupled partial differential equations which determine ϕ and ψ:

$$b_{22}\phi_{,1111} + (2b_{12} + a_{66})\phi_{,1122} + b_{11}\phi_{,2222} + (b_{25} + a_{46})\psi_{,112} + b_{15}\psi_{,222} = 0,$$
$$(b_{25} + a_{46})\phi_{,112} + b_{15}\phi_{,222} + a_{44}\psi_{,11} + b_{55}\psi_{,22} = -2\alpha - a_{35}B, \tag{2.11}$$

for $(x_1, x_2) \in D$, where $D = \{(x_1, x_2): -\frac{1}{2}a \le x_1 \le \frac{1}{2}a, -\frac{1}{2}b \le x_2 \le \frac{1}{2}b\}$. The modified constants $b_{\alpha\beta}$ introduced here are defined in terms of the $a_{\alpha\beta}$ by the formulae

$$b_{\alpha\beta} = a_{\alpha\beta} - \frac{a_{\alpha 3}a_{\beta 3}}{a_{33}}, \alpha, \beta = 1, 2, 5, \tag{2.12}$$

where there is no summation convention for Greek subscripts.

General expressions for the displacements u_i in the beam are obtained on integrating equations (2.4). Written out in full, these equations are

$$u_{1,1} = e_{11}, \quad u_{2,2} = e_{22}, \quad u_{3,3} = e_{33},$$
$$u_{2,3} + u_{3,2} = 2e_{23}, \quad u_{1,3} + u_{3,1} = 2e_{13},$$
$$u_{1,2} + u_{2,1} = 2e_{12}. \tag{2.13}$$

It now follows from equations $(2.13)_{3,4,5}$ and $(2.8)_1$ that

$$u_1 = -\tfrac{1}{2}a_{33}Ax_3^2 + x_3(2e_{13} - \overline{U}_{3,1}) + U_1$$
$$u_2 = -\tfrac{1}{2}a_{33}Bx_3^2 + x_3(2e_{23} - \overline{U}_{3,2}) + U_2$$
$$u_3 = a_{33}x_3(Ax_1 + Bx_2 + C) + \overline{U}_3 \tag{2.14}$$

where U_1, U_2 and $\overline{U}_3$ are arbitrary functions of x_1 and x_2. If we substitute these results into equations $(2.13)_{1,2}$ and equate coefficients of x_3 we obtain

$$e_{11} = U_{1,1}, \qquad e_{22} = U_{2,2},$$
$$(2e_{13} - \overline{U}_{3,1})_{,1} = (2e_{23} - \overline{U}_{3,2})_{,2} = 0, \tag{2.15}$$

whence

$$2e_{13} - \overline{U}_{3,1} = K_2(x_2), \qquad 2e_{23} - \overline{U}_{3,2} = K_1(x_1)$$

for arbitrary functions K_1, K_2. On making use of equations $(2.8)_2$ and $(2.13)_6$ we deduce that

$$K_1 = \alpha x_1 + v_2, \qquad K_2 = -\alpha x_2 + v_1 \tag{2.16}$$

where v_1 and v_2 are constants. The expressions (2.14) for the displacements may now be written in the general form

$$u_1 = -\tfrac{1}{2}a_{33}Ax_3^2 - \alpha x_2 x_3 + v_1 x_3 + U_1(x_1, x_2),$$
$$u_2 = -\tfrac{1}{2}a_{33}Bx_3^2 + \alpha x_1 x_3 + v_2 x_3 + U_2(x_1, x_2),$$
$$u_3 = a_{33}x_3(Ax_1 + Bx_2 + C) - v_1 x_1 - v_2 x_2 + U_3(x_1, x_2), \tag{2.17}$$

where the $U_i(x_1, x_2)$ satisfy

$$U_{1,1} = a_{13}(Ax_1 + Bx_2 + C) + b_{12}\phi_{,11} + b_{11}\phi_{,22} + b_{15}\psi_{,2},$$
$$U_{2,2} = a_{23}(Ax_1 + Bx_2 + C) + b_{22}\phi_{,11} + b_{12}\phi_{,22} + b_{25}\psi_{,2},$$
$$U_{3,2} = -\alpha x_1 - a_{44}\psi_{,1} - a_{46}\phi_{,12}, \tag{2.18}$$

and where $U_3 = \overline{U}_3 + v_1 x_1 + v_2 x_2$.

Our analysis has now reduced the bending and torsion problem for the beam to the solution of equations (2.11) for $\phi(x_1, x_2)$, $\psi(x_1, x_2)$. The stresses, strains and displacements in the beam then follow from equations (2.7), (2.9), (2.10) and (2.17). To complete the mathematical specification of the problem it is necessary to interpret the constants A, B, C, α, v_1, v_2 and find boundary conditions for the integration of equations (2.11).

(b) *Boundary conditions*

These are of two types, displacement conditions at the ends of the beam and the specification of forces and moments over the surface of the beam. Appropriate displacement conditions for a simply supported beam are

$$u_1 = u_2 = u_3 = 0, \quad \text{at} \quad x_1 = x_2 = 0, \qquad x_3 = 0, l. \tag{2.19}$$

On applying these conditions to equations (2.17) we obtain

$$U_1(0,0) = U_2(0,0) = U_3(0,0) = C = 0, \qquad v_1 = \tfrac{1}{2}a_{33}Al, \qquad v_2 = \tfrac{1}{2}a_{33}Bl, \tag{2.20}$$

whence the displacements u_i may be written

$$\begin{aligned} u_1 &= -\tfrac{1}{2}a_{33}Ax_3(x_3-l)-\alpha x_2x_3+U_1(x_1,x_2), \\ u_2 &= -\tfrac{1}{2}a_{33}Bx_3(x_3-l)+\alpha x_1x_3+U_2(x_1,x_2), \\ u_3 &= a_{33}Ax_1(x_3-\tfrac{1}{2}l)+a_{33}Bx_2(x_3-\tfrac{1}{2}l)+U_3(x_1,x_2). \end{aligned} \tag{2.21}$$

If we compare these expressions with typical bending and torsion solutions for an isotropic elastic beam, (see, for example, [4] equations (32.9), (34.3)) we see that A, B characterize bending of the beam about the x_2 and x_1 axes respectively, and α is its angle of twist per unit length about the x_3-axis. The function U_3 determines the warping, and U_1, U_2 the stretching of each cross-section D of the beam.

The long sides of the beam are assumed to be stress-free, leading to the following boundary conditions for t_{ij}:

$$\left.\begin{aligned} t_{11} = t_{12} = t_{13} = 0, &\quad \text{for } x_1 = \pm\tfrac{1}{2}a, \\ t_{21} = t_{22} = t_{23} = 0, &\quad \text{for } x_2 = \pm\tfrac{1}{2}b. \end{aligned}\right\} \quad 0 \le x_3 \le l. \tag{2.22}$$

We also assume that the stress distribution at the ends of the beam is equipollent to a bending moment M and twisting moment M_t. On balancing the moments of forces about the x_1, x_2 and x_3 axes at a cross-section D, we obtain†

$$\begin{aligned} &\int_D t_{33}x_2\,\mathrm{d}S = M, \qquad \int_D t_{33}x_1\,\mathrm{d}S = 0, \\ &\int_D (t_{23}x_1-t_{13}x_2)\,\mathrm{d}S = M_t, \end{aligned} \tag{2.23}$$

where it is understood that the integral over D is a surface integral and $\mathrm{d}S = \mathrm{d}x_1\,\mathrm{d}x_2$. When use is made of equations $(2.7)_3$ and $(2.8)_1$, the first two of these conditions become

$$\begin{aligned} BI_1-\frac{1}{a_{33}}\int_D (a_{13}t_{11}+a_{23}t_{22}+a_{35}t_{13})x_2\,\mathrm{d}S &= M, \\ AI_2-\frac{1}{a_{33}}\int_D (a_{13}t_{11}+a_{23}t_{22}+a_{35}t_{13})x_1\,\mathrm{d}S &= 0, \end{aligned} \tag{2.24}$$

where I_1, I_2 are the moments of inertia of D about the x_1 and x_2 axes respectively. Integrals of the type occurring here may be evaluated by using equations (2.5), (2.22) and the divergence theorem. For example,

$$\begin{aligned} \int_D t_{11}x_1\,\mathrm{d}S &= \int_D [x_1t_{11}+\tfrac{1}{2}x_1^2(t_{11,1}+t_{12,2})]\,\mathrm{d}S, \\ &= \int_D [(\tfrac{1}{2}x_1^2t_{11})_{,1}+(\tfrac{1}{2}x_1^2t_{12})_{,2}]\,\mathrm{d}S, \\ &= \int_{\partial D} \tfrac{1}{2}x_1^2(t_{11}n_1+t_{12}n_2)\,\mathrm{d}S = 0, \end{aligned}$$

† We adopt the sign convention that anticlockwise moments about an axis are positive. See Fig. 1.

where ∂D is the boundary of D, s is the arc length and n_1, n_2 the components of the outward normal to ∂D. The following results may also be established by similar arguments (see, for example, [2] equations (18.11)–(18.13))

$$\int_D t_{11}x_i \, dS = \int_D t_{22}x_i \, dS = 0, \qquad i = 1, 2,$$

$$\int_D t_{13}x_1 \, dS = 0, \qquad \int_D t_{13}x_2 \, dS = -\tfrac{1}{2}M_t. \tag{2.25}$$

Equations (2.24) now yield the conditions

$$A = 0, \qquad B = \frac{1}{I}\left(M - \frac{a_{35}}{2a_{33}}M_t\right), \tag{2.26}$$

where $I = I_1 = b^3a/12$. The constants A, B, C (if non-zero) are, therefore determined by the applied moments, the remaining constant α being the angle of twist of the beam.

We now substitute for ϕ and ψ in the stress conditions. On using equations (2.9), (2.22) and integrating around the contour ∂D we obtain the boundary conditions

$$\phi = \frac{\partial \phi}{\partial n} = \psi = 0 \quad \text{on } \partial D, \tag{2.27}$$

which replace (2.22), where $\partial/\partial n$ is the outward normal derivative on ∂D. Thus it follows from equations (2.9), $(2.23)_3$, (2.27) and the divergence theorem that

$$M_t = 2\int_D \psi \, dS. \tag{2.28}$$

Summary of mathematical formulation

The bending and torsion problems for a monoclinic elastic beam of rectangular cross-section are now reduced to the determination of ϕ and ψ from the partial differential equations (2.11) with the boundary conditions (2.27). The stress distribution and displacement then follow from equations (2.9), (2.10), (2.18), (2.21) with (2.26) and (2.28).

3. AN "OFF-AXIS" TRANSVERSELY ISOTROPIC BEAM

The main object of this paper is to derive formulae for determining the five elastic constants of a transversely isotropic elastic material from a combination of bending and torsion tests. In general it is not possible to obtain these constants by simple tests on specimens in which a principal axis coincides with the preferred direction. Thus a bending or torsion test must be performed on an "off-axis" specimen. The elastic constants $a_{\alpha\beta}$ will now be defined so that the analysis of Section 2 may be used for such tests.

We assume that the elastic beam depicted in Fig. 1 is composed of a transversely isotropic elastic material whose preferred direction is parallel to the $0x_1x_3$ plane and inclined at an angle $-\theta$ to the x_3-axis. For example, the beam could be fabricated from a fibre-reinforced material in which the fibres are aligned in layers parallel to the $0x_1x_3$ plane and inclined to the x_3-axis. We now consider a second set of Cartesian coordinate axes $0x_i'$ ($i = 1, 2, 3$) such that $0x_3'$ is the preferred direction of the elastic material and $0x_2'$ coincides with $0x_2$, as shown in Fig. 2. The elastic material is transversely isotropic

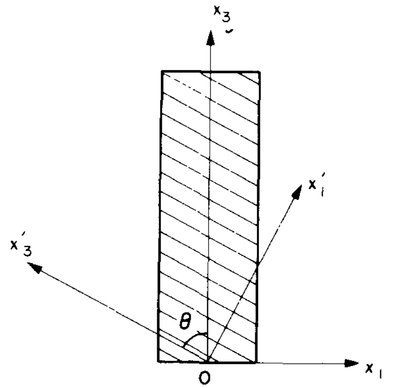

FIG. 2. The x_i and x_i' co-ordinate systems.

when referred to the x_i'-axes but it is considered to be monoclinic, with the single plane of symmetry $x_2 = 0$ when referred to the x_i-axes, the principal axes of the beam.

Let s_{ijkl}, a_{ijkl} be compliance tensors for the material when referred to the x_i' and x_i axes, respectively, and let r_{ij} be the coordinate transformation tensor for rotations about the x_2-axis from $0x_3'$ to $0x_3$, then $x_i = r_{ij}x_j'$ where

$$(r_{ij}) = \begin{pmatrix} \cos\theta, & 0, & -\sin\theta \\ 0, & 1, & 0 \\ \sin\theta, & 0, & \cos\theta \end{pmatrix} \tag{3.1}$$

and

$$a_{ijkl} = r_{im}r_{jn}r_{kp}r_{lq}s_{mnpq}. \tag{3.2}$$

These equations define the compliance tensor a_{ijkl} of the beam in terms of the compliances s_{ijkl} and the angle θ. We now revert to the two suffix notation for the compliances ([3], Section 1.3) so that $s_{ijkl} \to s_{\alpha\beta}$, $a_{ijkl} \to a_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 6$). The only five independent, non-zero compliances for a transversely isotropic material are s_{11}, s_{12}, s_{13}, s_{33}, s_{44}, the remainder of the $s_{\alpha\beta}$ satisfying

$$s_{\alpha\beta} = s_{\beta\alpha}, \quad s_{22} = s_{11}, \quad s_{23} = s_{13}, \quad s_{55} = s_{44}, \quad s_{66} = 2(s_{11} - s_{12}),$$
$$s_{14} = s_{15} = s_{16} = s_{24} = s_{25} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0. \tag{3.3}$$

On expanding equations (3.2), using the definitions (3.1) and the two suffix notation we obtain

$$
\begin{aligned}
a_{11} &= s_{11}\cos^4\theta+(2s_{13}+s_{44})\cos^2\theta\sin^2\theta+s_{33}\sin^4\theta,\\
a_{12} &= s_{12}\cos^2\theta+s_{13}\sin^2\theta,\\
a_{13} &= s_{13}+(s_{11}+s_{33}-2s_{13}-s_{44})\sin^2\theta\cos^2\theta,\\
a_{15} &= \sin\theta\cos\theta[2s_{11}\cos^2\theta+(2s_{13}+s_{44})(\sin^2\theta-\cos^2\theta)-2s_{33}\sin^2\theta],\\
a_{22} &= s_{11},\\
a_{23} &= s_{12}\sin^2\theta+s_{13}\cos^2\theta,\\
a_{25} &= 2(s_{12}-s_{13})\sin\theta\cos\theta,\\
a_{33} &= s_{11}\sin^4\theta+(2s_{13}+s_{44})\sin^2\theta\cos^2\theta+s_{33}\cos^4\theta,\\
a_{35} &= \sin\theta\cos\theta[2s_{11}\sin^2\theta+(2s_{13}+s_{44})(\cos^2\theta-\sin^2\theta)-2s_{33}\cos^2\theta],\\
a_{44} &= 2(s_{11}-s_{12})\sin^2\theta+s_{44}\cos^2\theta,\\
a_{46} &= (2s_{11}-2s_{12}-s_{44})\sin\theta\cos\theta,\\
a_{55} &= s_{44}+4(s_{11}+s_{33}-2s_{13}-s_{44})\sin^2\theta\cos^2\theta,\\
a_{66} &= 2(s_{11}-s_{12})\cos^2\theta+s_{44}\sin^2\theta,\\
a_{14} &= a_{16} = a_{24} = a_{26} = a_{34} = a_{36} = a_{45} = a_{56} = 0,
\end{aligned}
\tag{3.4}
$$

which define the elastic compliances $a_{\alpha\beta}$, referred to the principal axes of the beam, in terms of the five constants s_{11}, s_{12}, s_{13}, s_{33}, s_{44} and θ. The analysis of Section 2 with the $a_{\alpha\beta}$ defined as in (3.4) may now be used to describe bending and torsion of an "off-axis" transversely isotropic beam.

If a numerical or analytic solution were known for the boundary-value problem (2.11), (2.27) we could determine the deformation of any particular beam under specified bending and twisting moments. These results would have applications in engineering design with anisotropic materials. However, our main interest here is the determination of the elastic constants of the material from the observed deformation of the beam under known bending and twisting moments. This inverse problem is more difficult to solve. We see from equations (2.12) and (3.4) that the basic equations (2.11) are already quadratic in the $s_{\alpha\beta}$. An exact solution could be represented as a series whose coefficients are nonlinear functions of the $s_{\alpha\beta}$. A numerical solution is also of little direct use since the coefficients in the equations must be specified *a priori*.

We, therefore, consider the possibility of finding approximate solutions to the problem which will provide useful formulae for the $s_{\alpha\beta}$. Equilibrium problems in elasticity theory may be solved by the construction of a certain integral which attains its minimum value when the stress distribution in the body corresponds to an equilibrium state. The calculus of variations may then be used to determine this stress distribution. The main value of this variational formulation is the ease with which approximate solutions may be obtained, (see, for example, [4], Ch. 7). In the next section we construct this integral; approximate solutions to our problem are obtained in later sections. These solutions are then compared with a numerical solution to equations (2.11).

4. VARIATIONAL FORMULATION OF THE PROBLEM

Denoting the strain energy of the elastic beam shown in Fig. 1 as U and letting S be the end surfaces with outward normal n_i, ($n_1 = 0$, $n_2 = 0$, $n_3 = \pm 1$), we define the complementary energy

$$W = U - \int_S t_{ij} n_j u_i \, dS. \tag{4.1}$$

The theorem of minimum complementary energy ([4], Section 108) states that W has an absolute minimum when the stress tensor t_{ij} is that of the equilibrium elastic state of the beam. In deriving the principle it is assumed that t_{ij} satisfies the equilibrium equations (2.11) and the stress boundary conditions (2.22). The complementary energy W is then a minimum when the strains (2.7) also satisfy the compatibility conditions (2.6).

We now construct a simplified expression for W by making use of some results from Section 2. The strain energy U for the beam is defined by

$$U = \tfrac{1}{2} l \int_D t_{ij} e_{ij} \, dS, \tag{4.2}$$

which becomes, on using equations (2.7), (2.8), (with $A = C = 0$) and (2.12),

$$U = \tfrac{1}{2} l \int_D [b_{11} t_{11}^2 + b_{22} t_{22}^2 + a_{44} t_{23}^2 + b_{55} t_{13}^2 + a_{66} t_{12}^2 + 2b_{12} t_{11} t_{22} + 2b_{15} t_{11} t_{13} + 2b_{25} t_{22} t_{13} + 2a_{46} t_{12} t_{23} + a_{33} B^2 x_2^2] \, dS. \tag{4.3}$$

In order to evaluate the surface contribution to W we use expressions (2.21) for u_i, thus

$$\int_S t_{ij} n_j u_i \, dS = \int_D [t_{i3} u_i]_{x_3 = l} \, dS - \int_D [t_{i3} u_i]_{x_3 = 0} \, dS,$$
$$= l \int_D (-\alpha x_2 t_{13} + \alpha x_1 t_{23} + t_{33} B a_{33} x_2) \, dS.$$

On substituting for t_{33} we obtain

$$\int_S t_{ij} n_j u_i \, dS = l \int_D [\alpha x_1 t_{23} - (\alpha + a_{35} B) x_2 t_{13} + a_{33} B^2 x_2^2 - B x_2 (a_{13} t_{11} + a_{23} t_{22})] \, dS. \tag{4.4}$$

The complementary energy W now follows from equations (4.1), (4.3) and (4.4).

In order to apply the complementary energy principle, the admissible stresses t_{ij} must satisfy the equilibrium equations (2.5) and the stress boundary conditions (2.22). We, therefore, introduce the stress functions ϕ and ψ defined in equations (2.9). On substituting for t_{ij} in equations (4.3) and (4.4) and using $(2.23)_2$, $(2.25)_3$ and (2.28) we have

$$W = l \int_D F(\phi, \psi, x_1, x_2) \, dS, \tag{4.5}$$

where

$$F = \tfrac{1}{2} b_{11} (\phi_{,22})^2 + \tfrac{1}{2} b_{22} (\phi_{,11})^2 + \tfrac{1}{2} a_{66} (\phi_{,12})^2 + \tfrac{1}{2} a_{44} (\psi_{,1})^2 + \tfrac{1}{2} b_{55} (\psi_{,2})^2 + b_{12} \phi_{,11} \phi_{,22} + b_{15} \phi_{,22} \psi_{,2} + b_{25} \phi_{,11} \psi_{,2} + a_{46} \phi_{,12} \psi_{,1} - (2\alpha + a_{35} B) \psi - \tfrac{1}{2} a_{33} B^2 x_2^2. \tag{4.6}$$

The functional is minimized on the set of functions ϕ, ψ in D which satisfy the boundary conditions

$$\phi = \frac{\partial \phi}{\partial n} = \psi = 0, \quad \text{on} \quad \partial D. \tag{4.7}$$

This problem in the calculus of variations determines ϕ and ψ (and hence t_{ij}, e_{ij}, u_i), replacing the boundary-value problem (2.11), (2.27). It may be verified that (2.11) are the Euler equations for the variational problem (4.5), (4.6), and hence both formulations of the problem are equivalent.

We now provide a brief description of the Rayleigh–Ritz method for the approximate solution of variational problems, (e.g., Courant and Hilbert, [5], IV Section 2). If ϕ_i, ψ_i, $i = 1, \ldots, n$ are complete sets of functions in D which satisfy (4.7) on ∂D, we construct the functions

$$\phi_m^* = \sum_{i=1}^{m} a_i \phi_i, \qquad \psi_m^* = \sum_{i=1}^{m} b_i \psi_i, \qquad m \leq n, \tag{4.8}$$

where a_i, b_i are to be determined so that $W(\phi_m^*, \psi_m^*)$ is a minimum. On substituting from equation (4.8) into (4.5) and integrating, we obtain $W = W(a_1, \ldots, a_m, b_1, \ldots b_m)$. Parameters a_i, b_i which minimize W are now solutions of the algebraic equations

$$\frac{\partial W}{\partial a_i} = 0, \qquad \frac{\partial W}{\partial b_i} = 0, \qquad i = 1, \ldots, m. \tag{4.9}$$

If for some m, $W(\phi_m^*, \psi_m^*)$ is the absolute minimum of W, then $\phi = \phi_m^*$, $\psi = \psi_m^*$ are the required solutions to the variational problem. It is doubtful whether the exact solution could be constructed in this way. However, a good approximation to it is obtained when a_i, b_i are chosen to minimize W for a given choice of ϕ_i, ψ_i and m. It has been shown for special forms of the energy integral (4.5), (4.6) (for example, when the material is isotropic) that

$$\lim_{n \to \infty} W(\phi_n^*, \psi_n^*) = W(\phi, \psi)$$

and under certain conditions it then follows that the series $\{\phi_n^*, \psi_n^*\}$ converges to the exact solution (ϕ, ψ). Some results of this type may be found in Kantorovitch and Krylov ([6], IV, Section 4) with estimates for the order of convergence of the series. In the absence of a specific convergence proof for the integral (4.5) we shall use a numerical solution to the partial differential equations (2.11) as a test for the validity of approximate solutions obtained by the Rayleigh–Ritz method.

5. APPROXIMATE SOLUTIONS TO THE VARIATIONAL PROBLEM

The numerical solution of equations (2.11) and the integrations (4.5) are facilitated if ϕ and ψ are expressed in dimensionless form. We, therefore, introduce dimensionless coordinates X_1, X_2, stress functions Φ, Ψ and elastic constants $A_{\alpha\beta}$, $B_{\alpha\beta}$ defined as follows:

$$X_1 = \frac{2x_1}{a}, \qquad X_2 = \frac{2x_2}{b},$$

$$\Phi = \frac{8a_{33}}{b^3(2\alpha + a_{35}B)}\phi, \qquad \Psi = \frac{4a_{33}}{b^2(2\alpha + a_{35}B)}\psi,$$

$$A_{\alpha\beta} = \frac{a_{\alpha\beta}}{a_{33}}, \qquad B_{\alpha\beta} = \frac{b_{\alpha\beta}}{a_{33}} = A_{\alpha\beta} - A_{\alpha 3} A_{\beta 3}. \tag{5.1}$$

By direct substitution into equations (2.11) and (2.27) we obtain the following boundary-value problem for the determination of Φ and Ψ:

$$\lambda^4 B_{22}\Phi_{,1111}+\lambda^2(2B_{12}+A_{66})\Phi_{,1122}+B_{11}\Phi_{,2222}+\lambda^2(B_{25}+A_{46})\Psi_{,112}+B_{15}\Psi_{,222}=0,$$

$$\lambda^2(B_{25}+A_{46})\Phi_{,112}+B_{15}\Phi_{,222}+\lambda^2 A_{44}\Psi_{,11}+B_{55}\Psi_{,22}=-1, \tag{5.2}$$

for $(X_1, X_2)\in D^*$,

$$\Phi=\frac{\partial\Phi}{\partial n}=\Psi=0,\quad \text{for } (X_1, X_2)\in \partial D^*, \tag{5.3}$$

where

$$D^*=\{(X_1,X_2): -1\leq X_1, X_2\leq 1\},\quad \lambda=b/a$$

and $\Phi_{,i}$ now denotes $\partial\Phi/\partial X_i$. We do not record dimensionless forms for stresses and displacements since these follow directly from equation (5.1) and the definitions of Section 2.

The variational formulation of the problem may also be cast in dimensionless form. Let W^* be the dimensionless complementary energy

$$W^*=\frac{16a_{33}}{ab^3 l(2\alpha+a_{35}B)^2}(W+\tfrac{1}{2}a_{33}IB^2 l), \tag{5.4}$$

then Φ and Ψ minimize the functional

$$W^*=\int_{D^*}F^*\,\mathrm{d}S, \tag{5.5}$$

where

$$F^*=\tfrac{1}{2}B_{11}(\Phi_{,22})^2+\tfrac{1}{2}\lambda^4 B_{22}(\Phi_{,11})^2+\tfrac{1}{2}\lambda^2 A_{66}(\Phi_{,12})^2+\tfrac{1}{2}\lambda^2 A_{44}(\Psi_{,1})^2+\tfrac{1}{2}B_{55}(\Psi_{,2})^2$$
$$+\lambda^2 B_{12}\Phi_{,11}\Phi_{,22}+B_{15}\Phi_{,22}\Psi_{,2}+\lambda^2 B_{25}\Phi_{,11}\Psi_{,2}+\lambda^2 A_{46}\Phi_{,12}\Psi_{,1}-\Psi, \tag{5.6}$$

subject to conditions (5.3).

Our analysis has been mainly concerned with the determination of Φ and Ψ for the bending and torsion of anisotropic beams. However, we do not require a detailed knowledge of the stress distribution within the beam in order to interpret experimental results. In classical bending tests the deflection of the midpoint of the beam is measured when a known bending moment is applied and in torsion tests the ratio of applied twisting moment to angle of twist is calculated. We now derive corresponding relations for anisotropic beams.

Equations (2.28) and (5.1) provide the following relation between twisting moment M_t and angle of twist α

$$M_t=\frac{3I(2\alpha+a_{35}B)M^*}{2a_{33}}, \tag{5.7}$$

where

$$M^*=\int_{D^*}\Psi\,\mathrm{d}S \tag{5.8}$$

and we recall that

$$B = (M - \tfrac{1}{2}A_{35}M_t)/I.$$

Referring to equations (2.21) for the displacements in the beam, we obtain for v, the x_2-component of displacement for the midpoint of the upper face of the beam,

$$v \equiv u_2(0, -b/2, l/2) = \tfrac{1}{8}Ba_{33}l^2 + U_2(0, -b/2). \tag{5.9}$$

The first term on the right side of this equation is the displacement of the midpoint of the beam $(0, 0, l/2)$. The term $U_2(0, -b/2)$ is the correction for the change in thickness of the beam as it is bent and twisted, and is expected to be small. On integrating equation $(2.18)_2$ and using condition $(2.20)_2$ we obtain

$$U_2(0, -b/2) = \tfrac{1}{8}a_{23}Bb^2 + \tfrac{1}{2}b_{22}\int_0^{-b/2} \phi_{,11}(0, x)\,dx - \tfrac{1}{2}b_{12}\phi_{,2}(0, 0) - b_{25}\psi(0, 0), \tag{5.10}$$

which when expressed in dimensionless form gives for v

$$v = \tfrac{1}{8}Ba_{33}l^2\left(1 + \frac{b^2}{l^2}A_{23}\right) + \tfrac{1}{4}b^2(2\alpha + a_{35}B)\left[\lambda^2 B_{22}\int_0^{-1} \Phi_{,11}(0, x)\,dx - B_{12}\Phi_{,2}(0, 0) - B_{25}\Psi(0, 0)\right]. \tag{5.11}$$

Equations (5.7) and (5.11) are the main results for the interpretation of bending and torsion experiments for anisotropic beams. We observe that Φ and Ψ are not needed for all $(X_1, X_2) \in D^*$, the main requirement being the evaluation of M^* and the derivatives which occur in (5.11).

The coupling which exists between bending and torsion is shown in these equations. For example, in a torsion test with $M = 0$, $M_t \neq 0$, we have $B = -\tfrac{1}{2}A_{35}M_t/I \neq 0$. It then follows from equation (5.11) that the beam bends about the x_1-axis and twists about the x_3-axis. Similarly, in a bending test with $M \neq 0$, $M_t = 0$, equation (5.7) shows that the beam twists through an angle $\alpha = -\tfrac{1}{2}a_{35}M/I \neq 0$. Thus, provided $a_{35} \neq 0$, these two modes of deformation are coupled together. We note that for a beam of transversely isotropic material $a_{35} = 0$ when $\theta = 0, \pi/2$ (see equation (3.4)). Use will be made of this observation in Section 6.

The presence of this coupling leads to experimental difficulties in bending and torsion tests. For example, in a bending test the beam may twist off the supports. We, therefore, examine the possibility of applying both bending and twisting moments so that the beam is in a state of pure bending or pure torsion.

(a) *Bending without twisting*

The beam does not twist when M and M_t are chosen so that $\alpha = 0$. On setting $\alpha = 0$ in equation (5.7) and solving for M_t we obtain

$$M_t = \frac{3A_{35}M^*M}{2 + \tfrac{3}{2}A_{35}^2M^*}, \tag{5.12}$$

and hence

$$B = \frac{M}{I(1 + \tfrac{3}{4}A_{35}^2M^*)}. \tag{5.13}$$

The deflection v now becomes

$$v = \frac{Ma_{33}l^2}{8I(1+\frac{3}{4}A_{35}^2M^*)}\left\{1+\frac{b^2F}{l^2}\right\}, \tag{5.14}$$

where the correction term F is given by

$$F = A_{23}+2A_{35}\left[\lambda^2B_{22}\int_0^{-1}\Phi_{,11}(0,x)\,\mathrm{d}x - B_{12}\Phi_{,2}(0,0) - B_{25}\Psi(0,0)\right]. \tag{5.15}$$

If we assume that $b^2F/l^2 \ll 1$, as justified later, then equation (5.14) is very similar to the classical deflection formula for isotropic beams, namely

$$v = \frac{Ml^2}{8IE},$$

where E $(= 1/a_{33})$ is Young's modulus. Thus, to eliminate twisting in a bending test the moment M_t given by equation (5.12) must be applied to the beam. When twisting is prevented, we observe that the effective Young's modulus for the beam is increased by a factor $1+3A_{35}^2M^*/4$, provided b^2F/l^2 is negligible. Experimentally we apply a bending moment to the beam which is so clamped that it cannot twist; the moment M_t is then applied implicitly at the supports. It is not necessary to measure M_t nor to apply it explicitly.

(b) *Torsion without bending*

On examining expressions (2.21) for the displacements, with $A = 0$, we see that if $B = 0$ the beam is in torsion about the x_3-axis, the terms $U_i(x_1, x_2)$ representing a warping and stretching of the cross-sections D. Thus, when $B = 0$ the beam does not bend and from equations (2.26) and (5.7) we obtain

$$M_t = \frac{3I\alpha M^*}{a_{33}}, \tag{5.16}$$

$$M = \tfrac{1}{2}A_{35}M_t. \tag{5.17}$$

Here the expression for M is the bending moment applied implicitly at the supports. As in (a), it is only necessary to clamp the beam to prevent bending. We note that M^* in equation (5.16) will depend on the $a_{\alpha\beta}$, thus a_{33} is not a shear compliance. If the beam is allowed to bend freely then $M = 0$ and from equation (5.7) we obtain

$$M_t = \frac{3I\alpha M^*}{a_{33}(1+\frac{3}{4}A_{35}^2M^*)}. \tag{5.18}$$

On comparing equations (5.16) and (5.18) we see that the torsional rigidity of the beam is increased by a factor $1+3A_{35}^2M^*/4$ when bending is prevented.

Equations (5.14), (5.16) are generalizations of classical bending and torsion formulae for anisotropic beams. They are the basis for the series of experiments described in Section 6. We also describe in Section 6 the physical significance of the increased stiffness predicted here. The analysis of (a) and (b) shows that Φ and Ψ are only required in the evaluation of M^* and to check that b^2F/l^2 is negligible.

Numerical results

We now present numerical results for M^* and F obtained from the solution of equations (5.2) by a collocation method. Details of the calculations leading to these results are given in [7]. We consider an "off-axis", transversely isotropic beam made from a typical carbon fibre-epoxy resin material with elastic compliances

$$s_{11} = 13{\cdot}3\times 10^{-11}\,m^2/N, \qquad s_{33} = 0{\cdot}418\times 10^{-11}\,m^2/N,$$
$$s_{12} = -7\times 10^{-11}\,m^2/N, \qquad s_{13} = -0{\cdot}113\times 10^{-11}\,m^2/N,$$
$$s_{44} = 17{\cdot}4\times 10^{-11}\,m^2/N. \tag{5.19}$$

This data is taken from experimental results of Markham [1]. By allowing the fibre angle θ to vary, and using equations (3.4) and (5.19) to define the $a_{\alpha\beta}$, we are able to consider a class of monoclinic beams. It follows that M^* and F are functions of s_{11}, s_{12}, s_{13}, s_{33}, s_{44}, θ and $\lambda\,(= b/a)$ only. In Table 1 we present computed values of M^* for a material with compliances (5.19) and a range of values of θ and λ and in Table 2 we give corresponding results for F.

The results shown in Table 1 are now assumed to be exact values of M^* for the class of beams under consideration. In the final part of this section we shall derive some simple approximations to M^* whose validity will be tested by comparison with Table 1. From Table 2 we can estimate the error involved in neglecting the term b^2F/l^2 in the bending formula (5.14). It is clear that this error diminishes as the length of the beam is increased. The percentage error is less than n when

$$l > 10b\left(\frac{|F|}{n}\right)^{\frac{1}{2}}. \tag{5.20}$$

We see from Table 2 that a typical value of $|F|$ for the material under consideration is $|F| = 0{\cdot}5$. Our bending formula is then accurate to within 0·1 per cent provided $l > 22{\cdot}4b$. Thus, for a typical long beam with dimensions 10 cm × 1 cm × 0·5 cm we would expect the bending formula (5.14) with $F \equiv 0$ to be accurate to within about 0·1 per cent.

TABLE 1

λ	$\theta°$	0°	15°	30°	45°	60°	75°	90°
	M^*	0·0135	0·0682	0·172	0·236	0·258	0·26	0·259
1	M_1	0·0133	0·0666	0·166	0·228	0·251	0·255	0·255
	M_3	0·0135	0·0681	0·172	0·236	0·258	0·26	0·259
	M^*	0·022	0·157	0·429	0·551	0·568	0·553	0·544
1/2	M_1	0·0214	0·155	0·423	0·542	0·559	0·545	0·536
	M_3	0·0219	0·156	0·427	0·548	0·565	0·552	0·544
	M^*	0·027	0·239	0·708	0·855	0·837	0·793	0·774
1/4	M_1	0·0251	0·231	0·688	0·826	0·805	0·76	0·741
	M_3	0·0269	0·239	0·706	0·853	0·835	0·791	0·773
	M^*	0·03	0·295	0·9	1·06	1·01	0·947	0·921
1/10	M_1	0·0264	0·268	0·835	0·968	0·918	0·855	0·830
	M_3	0·0293	0·292	0·898	1·05	1·0	0·935	0·909

TABLE 2

λ	$\theta°$	0°	15°	30°	45°	60°	75°	90°
	F	−0·27	−0·28	−0·25	−0·39	−0·45	−0·51	−0·53
1	N	1	1·72	1·56	1·25	1·08	1·02	1
	N_1	1	1·72	1·55	1·24	1·08	1·02	1
	F	−0·27	−0·12	0·04	−0·23	−0·48	−0·51	−0·53
1/2	N	1	2·65	2·40	1·58	1·18	1·03	1
	N_1	1	2·63	2·38	1·57	1·17	1·03	1
	F	−0·27	0·15	0·53	−0·17	−0·31	−0·51	−0·53
1/4	N	1	3·52	3·31	1·89	1·26	1·05	1
	N_1	1	3·44	3·25	1·88	1·25	1·04	1
	F	−0·27	0·45	−0·07	−0·35	−0·36	−0·49	−0·53
1/10	N	1	4·12	3·97	2·10	1·31	1·05	1
	N_1	1	3·82	3·73	2·01	1·28	1·05	1

Approximations to M^*

We now estimate M^* using the Rayleigh–Ritz method described in Section 4. Since the applications of this paper are to "off-axis" transversely isotropic beams we shall be mainly interested in approximate formulae for M^* which are valid uniformly as θ varies and for a range of values of λ. However, the results obtained will apply to any monoclinic material.

First trial functions

Examination of equations (5.2) shows that $\Phi \equiv 0$ is a solution if and only if there exists a scalar μ such that

$$B_{25}+A_{46} = \mu A_{44}, \qquad B_{15} = \mu B_{55}.$$

This condition is not satisfied in general. However, in the special case $\theta = 0, \pi/2$ it follows from the definitions of $A_{\alpha\beta}$, $B_{\alpha\beta}$ that $B_{25} = A_{46} = B_{15} = 0$, hence the above condition is satisfied identically with $\mu = 0$. Thus $\Phi \equiv 0$ when $\theta = 0, \pi/2$, and it is reasonable to suppose that Φ remains small when θ is close to 0 or $\pi/2$. We therefore, take as our first pair of trial functions

$$\Phi = 0, \qquad \Psi = K(X_1^2-1)(X_2^2-1), \tag{5.21}$$

which satisfy the boundary conditions (5.3). This equation is of the form (4.8) with $m = 1$. The parameter K is now chosen to minimize the integral (5.5). On substituting (5.21) into (5.6) and evaluating the integral (5.5) we obtain

$$W^* = \frac{16K}{9}\left[\frac{4K}{5}(\lambda^2 A_{44}+B_{55})-1\right].$$

$W^*(K)$ is a minimum when $\mathrm{d}W^*/\mathrm{d}K = 0$, that is for

$$K = \frac{5}{8(\lambda^2 A_{44}+B_{55})}.$$

Thus

$$\Phi = 0, \qquad \Psi = \frac{5(X_1^2-1)(X_2^2-1)}{8(\lambda^2 A_{44}+B_{55})}, \tag{5.22}$$

are our first approximations to Φ and Ψ. Using equation (5.8) we obtain the following estimate M_1 for M^*

$$M_1 = \frac{10}{9(\lambda^2 A_{44}+B_{55})}. \tag{5.23}$$

This first trial function, therefore, provides a simple formula for M^* involving only two out of the seven coefficients occurring in the integral (5.5). M_1 is compared with M^* in Table 1 for a range of values of θ and λ. Reference to Table 1 shows that M_1 approximates M^* uniformly for all θ but the error increases as λ decreases from about 2 per cent at $\lambda = 1$ to 10 per cent at $\lambda = 0{\cdot}1$.

This error is too great for use with the torsion formula (5.16), except for obtaining rough estimates. However, in the bending formula (5.14) M^* occurs in the form $N = 1+3A_{35}^2M^*/4$. In Table 2 we compare $N_1 = 1+3A_{35}^2M_1/4$ with computed values of N. We see that there is close agreement for $\lambda = 1$, 0·5, 0·25 except for $\theta = 15°$, 30°. Thus, equation (5.14) with M^* given by (5.23) should suffice for the interpretation of "off-axis" bending tests.

Second trial functions

In order to examine the effect of neglecting Φ, we take as our second trial functions

$$\Phi = L_1 X_2(X_1^2-1)^2(X_2^2-1)^2, \qquad \Psi = L_2(X_1^2-1)(X_2^2-1). \tag{5.24}$$

These functions satisfy the boundary conditions (5.3) and Φ has been taken to be even in X_1 and odd in X_2 (see equations (5.2)). On substituting equations (5.24) into (5.5) and (5.6) we obtain $W^* = W^*(L_1, L_2)$. Choosing L_1 and L_2 to satisfy the linear algebraic equations $\partial W^*/\partial L_1 = \partial W^*/\partial L_2 = 0$ we find that

$$L_1 = \frac{-5R}{2(4PQ-R^2)}, \quad L_2 = \frac{5P}{2(4PQ-R^2)}, \tag{5.25}$$

where

$$P = \tfrac{8}{7}\left[\tfrac{5819}{7}B_{11}+\tfrac{256}{105}\lambda^2(2B_{12}+A_{66})+\tfrac{3098}{55}\lambda^4 B_{12}\right],$$

$$Q = \lambda^2 A_{44}+B_{55},$$

$$R = \frac{32\lambda^2(B_{25}+A_{46})}{35}. \tag{5.26}$$

If M_2 is a second approximation to M^* we obtain from (5.8) and (5.24)

$$M_2 = \frac{40P}{9(4PQ-R^2)}. \tag{5.27}$$

Values of M_2 were calculated from equation (5.27). However they agreed, to the first three significant figures, with values given for M_1 in Table 1 for all θ and λ. The reason for this became clear when P, Q and R were examined, since it was found that $R^2 \ll PQ$ and equation (5.27) could be approximated by

$$M_2 \simeq \frac{40P}{36PQ} = \frac{10}{9Q} \equiv M_1.$$

It was also found that $L_2 > 10^3 L_1$, thus $\Phi \ll \Psi$ for $(X_1, X_2) \in D^*$ and $\Phi \equiv 0$ is clearly a good approximation to the exact solution.

Third trial functions

The above discussion suggests that to improve our estimate for M^*, we should set $\Phi = 0$, and allow Ψ to have more free parameters. Since Ψ is an even function of X_1 and X_2 we take

$$\Phi = 0, \qquad \Psi = (X_1^2 - 1)(X_2^2 - 1)(K_1 + K_2 X_1^2 + K_3 X_2^2). \tag{5.28}$$

Following the method outlined in Section 4 we obtain a set of three algebraic equations for K_1, K_2 and K_3 whose solution is

$$\begin{aligned} K_1 &= 35(9\lambda^4 A_{44}^2 + 9B_{55}^2 + 130\lambda^2 A_{44} B_{55})/T, \\ K_2 &= 105 B_{55}(\lambda^2 A_{44} + 9B_{55})/T, \\ K_3 &= 105\lambda^2 A_{44}(9\lambda^2 A_{44} + B_{55})/T, \end{aligned} \tag{5.29}$$

where

$$T = 16(\lambda^2 A_{44} + B_{55})[45(\lambda^4 A_{44}^2 + B_{55}^2) + 464\lambda^2 A_{44} B_{55}].$$

If M_3 is a third approximation to M^* then it follows from equations (5.8), (5.28) and (5.29) that

$$M_3 = \frac{56(9\lambda^4 A_{44}^2 + 9B_{55}^2 + 82\lambda^2 A_{44} B_{55})}{9(\lambda^2 A_{44} + B_{55})[45(\lambda^4 A_{44}^2 + B_{55}^2) + 464\lambda^2 A_{44} B_{55}]}. \tag{5.30}$$

M_3 is compared with M_1 and M^* in Table 1. We see that for $\lambda = 1$ the error involved in using M_3 for M^* is less than 0·1 per cent, rising to 0·5 per cent for $\lambda = 0{\cdot}25$, 0·5 and 1 per cent for $\lambda = 0{\cdot}1$. We can therefore use M_3 for the interpretation of "off-axis" torsion tests.

More sophisticated trial functions could be constructed using the methods described above. These higher order approximations may be needed if accurate values for the stresses in the beam are required, since derivatives of Φ and Ψ enter the calculation of the stresses. This aspect of the solution does not concern us in this paper.

We have shown that $\Phi \equiv 0$ is an approximate solution to the boundary value problem (5.2), (5.3). Thus, to the same degree of approximation, the stress components t_{11}, t_{12}, t_{22} are negligible compared with t_{33}, t_{13}, t_{23} in our original formulation. This assumption could not have been made *a priori* since the omission of t_{11}, t_{12}, t_{22} would lead to an over-determined problem. However, we see that if $\Phi \equiv 0$ and equation $(5.2)_1$ is neglected we obtain the simplified boundary value problem

$$\lambda^2 A_{44} \Psi_{,11} + B_{55} \Psi_{,22} = -1, \qquad (X_1, X_2) \in D^*, \tag{5.31}$$

$$\Psi = 0, \qquad (X_1, X_2) \in \partial D^*, \tag{5.32}$$

which is well-posed. It is clear from our earlier discussion that the solution to equations (5.31), (5.32) will be a reasonable approximation to the exact solution of the full equations. This observation provides a useful starting point for approximate analytic solutions to problems of this type, and shows that the general formulation given in Section 2 may be simplified considerably for the solution to practical problems.

6. SUMMARY OF RESULTS AND PROPOSED EXPERIMENTAL PROGRAMME

In this section we provide a summary of the main results and describe a sequence of experiments to determine the five compliances of a transversely isotropic elastic material.

The beam shown in Fig. 1 is subjected to a constant bending moment M as in a four-point bending test, and supported at each end so that twisting is prevented. The deflection v of the midpoint of the upper surface is

$$v = \frac{a_{33}Ml^2}{8I(1 + 3a_{35}^2 M^*/4a_{33}^2)}, \tag{6.1}$$

where

$$M^* = \frac{10a_{33}}{9(\lambda^2 a_{44} + b_{55})}, \tag{6.2}$$

$b_{55} = a_{55} - a_{35}^2/a_{33}$, $\lambda = b/a$, $I = ab^3/12$, and $a_{\alpha\beta}$ are the compliances of a monoclinic elastic material; they may also be expressed in terms of the five compliances of a transversely isotropic material and the "fibre angle" θ, (see (3.4)). Equations (6.1) and (6.2) are a good approximation for long beams ($l > 20b$) with $1 \geq b/a > 1/4$.

If the same beam is now twisted about the x_3-axis but constrained by the supports not to bend, then the angle of twist α is related to the twisting moment M_t by

$$M_t = \frac{3I\alpha M^*}{a_{33}}, \tag{6.3}$$

where equation (6.2) provides a first approximation to M^*, but a better one is

$$M^* = \frac{56a_{33}[9\lambda^4 a_{44}^2 + 9b_{55}^2 + 82\lambda^2 a_{44}b_{55}]}{9(\lambda^2 a_{44} + b_{55})[45(\lambda^4 a_{44}^2 + b_{55}^2) + 464\lambda^2 a_{44}b_{55}]}. \tag{6.4}$$

These equations hold for a beam of monoclinic elastic material. However, since they only involve the compliances a_{33}, a_{44}, a_{55}, a_{35}, it is not possible to determine all thirteen elastic constants for a monoclinic beam by bending and torsion tests alone. Referring to equations (3.4), we see that a_{33}, a_{44}, a_{55}, a_{35} contain all five constants of a transversely isotropic material s_{11}, s_{12}, s_{13}, s_{33}, s_{44}, and the angle θ. Thus equations (6.2) and (6.4) relate the five compliances and the angle θ, and by varying θ it is possible to determine these constants using only bending and torsion tests. We now describe two possible series of experiments for the $s_{\alpha\beta}$, which make use of three specimens in which θ takes the values 0, $\pi/2$, $\pi/4$. These will be referred to as specimens I, II and III respectively.

Specimen I

On setting $\theta = 0$ in equations (3.4) we obtain

$$a_{33} = s_{33}, \qquad a_{44} = a_{55} = s_{44}, \qquad a_{35} = 0, \qquad a_{23} = s_{13}, \tag{6.5}$$

hence (6.1) and (6.3) yield

$$v = \frac{Ml^2 s_{33}}{8I}, \tag{6.6}$$

$$M_t = \frac{56I\alpha(9+82\lambda^2+9\lambda^4)}{3s_{44}(1+\lambda^2)[45(1+\lambda^4)+464\lambda^2]}. \tag{6.7}$$

Thus, a simple bending and torsion tests for specimen I determine s_{33} and s_{44} explicitly. We note that since $a_{35} = 0$, there is no coupling between bending and torsion.

Specimen II

When $\theta = \pi/2$, it follows from equations (3.4) that

$$a_{33} = s_{11}, \quad a_{44} = 2(s_{11}-s_{12}) = s_{66}, \quad a_{55} = s_{44}$$
$$a_{35} = 0, \quad a_{23} = s_{12}. \tag{6.8}$$

Therefore, (6.1) and (6.3) become

$$v = \frac{Ml^2 s_{11}}{8I}, \tag{6.9}$$

$$M_t = \frac{56I\alpha(9+82\mu+9\mu^2)}{3s_{44}(1+\mu)[45(1+\mu^2)+464\mu]}, \tag{6.10}$$

where

$$\mu = \frac{\lambda^2 s_{66}}{s_{44}} = \frac{2\lambda^2(s_{11}-s_{12})}{s_{44}}. \tag{6.11}$$

A simple bending test determines s_{11} explicitly and s_{12} may be found from a torsion test by solving equation (6.10) for μ.

The four experiments described above enable $s_{11}, s_{33}, s_{12}, s_{44}$ to be calculated. In order to determine the fifth constant s_{13} it is necessary to perform *either* an "off-axis" bending *or* an "off-axis" torsion test. We analyse both these tests for specimen III.

Specimen III

On substituting equation (6.2) into (6.1) and (6.3) we obtain the formulae

$$v = \frac{Ml^2 a_{33}}{8I}\left\{1 - \frac{5a_{35}^2}{6a_{33}(\lambda^2 a_{44}+a_{55})-a_{35}^2}\right\}, \tag{6.12}$$

$$M_t = \frac{10I\alpha a_{33}}{3a_{33}(\lambda^2 a_{44}+a_{55})-3a_{35}^2}, \tag{6.13}$$

as first approximations to v and M_t. Referring again to equations (3.4) with $\theta = \pi/4$ provides the relations

$$a_{33} = \tfrac{1}{4}(s_{11}+s_{33}+s_{44}+2s_{13}), \quad a_{44} = \tfrac{1}{2}(s_{44}+s_{66}),$$
$$a_{55} = s_{11}+s_{33}-2s_{13}, \quad a_{35} = \tfrac{1}{2}(s_{11}-s_{33}),$$

which when substituted into either of equations (6.12) or (6.13) enable s_{13} to be found. As noted previously a better approximation to v and M_t is obtained on using equation (6.4) instead of (6.2).

We now describe an alternative sequence of experiments using specimens I and II only. These experiments require an accurate measurement of v and are suitable for short beams.

Specimen I

On substituting from equations (6.5) into (5.14) and (5.15) we obtain the exact deflection formula

$$v = \frac{M(l^2 s_{33} + b^2 s_{13})}{8I}. \tag{6.14}$$

In the bending analysis described above, we assumed that $l > 20b$ hence the second term in this formula could be neglected. For short beams this is not the case, and we may use this formula to calculate s_{33} and s_{13}. If the bending rig is designed so that the effective length of the beam may be varied, then on measuring the deflection for different lengths of beam we obtain s_{13} and s_{33} by plotting v against l^2.

Specimen II

In this case the formula corresponding to (6.14) is

$$v = \frac{M(l^2 s_{11} + b^2 s_{12})}{8I}, \tag{6.15}$$

from which s_{11} and s_{12} are determined by the method described above.

The fifth constant s_{44} is obtained from a torsion test with specimen I (see equation (6.7)). A torsion test with specimen II is not now required. This second set of experiments seems to provide an easier method for determining the $s_{\alpha\beta}$. However end effects, discussed below, may be significant in beams short enough for the use of equations (6.14) and (6.15) and the results therefore require careful experimental investigation.

When the axes of material symmetry coincide with the beam axes, the torsion and flexure problems are considerably simplified and there are alternative solutions in the literature. In some cases, such as the deflection formulae (6.6) and (6.9), our results coincide with those given elsewhere (see [3], Section 4.3). In other cases we have provided a systematic approximation to a more complicated solution. Thus Hearman ([3], Section 4.4) gives an infinite series solution for torsion of a rectangular bar which in our analysis is replaced by equation (6.7). Flexure of "off-axis" plates has previously been discussed by Whitney and Dauksys [8] who showed that shear coupling could cause the plate to twist off the supports, making the measurement of material properties difficult. Our more detailed analysis of this phenomenon suggests that "off-axis" flexure tests are possible provided twisting is prevented at the supports.

One of our main assumptions, which is used widely in isotropic elasticity theory, is that St. Venant's principle holds so that clamping effects may be neglected. In their paper on off-axis tension tests of elastic composites Pagano and Halpin [9] attempt to analyse the effect of end constraints on the stress field. They show that conventional clamping devices may perturb the assumed state of uniform tensile stress in the beam causing shear and bending stresses to be present, which casts some doubt on our assumption that end effects need not be considered here. It is probable that our assumption is valid for long enough beams, and this is supported by the experimental evidence in [9] where a uniform tensile strain was observed in the central region of the beam with a length/width ratio of six. However it is impossible to ascribe boundary conditions which model precisely the clamping method used in an experiment, making it necessary to solve an idealized problem. The only test for the validity of the assumptions is whether the final results agree with experiment. Thus an experimentalist, wishing to use our results to determine material properties would first need to perform a series of experiments to test the range of validity of the formulae. For example, if end effects are significant in a flexure test, the deflection v would not be proportional to l^2 as predicted by (6.1), and this could be investigated by measuring v for a range of values of l.

Viscoelastic beams

The analysis of this paper may also be used to determine the frequency dependent compliances of anisotropic viscoelastic beams deformed by time-harmonic bending or twisting moments. The required results follow immediately from the correspondence between the Fourier transforms of the governing elastic and viscoelastic equations. Details of this correspondence are contained in [10]. Thus, when a viscoelastic beam undergoes forced steady bending or torsional oscillations with frequency ω, the deflection and angle of twist are given by the elastostatic equations (6.1)–(6.4) with the substitutions†

$$a_{\alpha\beta} \to a^*_{\alpha\beta}(\omega), \qquad s_{\alpha\beta} \to s^*_{\alpha\beta}(\omega), \qquad M_t \to M_t e^{i\omega t},$$
$$M \to M e^{i\omega t}, \qquad v \to v(t), \qquad \alpha \to \alpha(t), \tag{6.16}$$

where $a^*_{\alpha\beta}(\omega)$, $s^*_{\alpha\beta}(\omega)$ are the complex frequency dependent compliances of a viscoelastic material with the appropriate symmetries, and $M_t \exp(i\omega t)$, $M \exp(i\omega t)$ are the applied moments. Since the elastic solution is independent of time, it follows that the results derived from the correspondence principle apply only to quasi-static viscoelastic deformations where the frequency ω is low enough for vibratory inertia to be neglected. A condition realized in practice when $\omega \ll 1/T$, where T is a typical travel time for body waves in the beam.

The complex compliances $s^*_{\alpha\beta}(\omega)$ for a transversely isotropic viscoelastic material may now be obtained by performing a similar series of experiments to those described above and using the correspondence (6.16) in equations (6.6), (6.7), (6.9), (6.10), (6.12), (6.13). We shall briefly describe one such experiment—the bending test for specimen I. We assume that a bending moment of constant amplitude M and frequency ω is applied to the beam. The amplitude $\hat{v}$ and phase lag $\varepsilon(\omega)$ of the displacement $v(t)$ are measured when a steady state has been reached. On setting

$$v(t) = \hat{v}\, e^{i(\omega t - \varepsilon)}, \qquad M = M\, e^{i\omega t}, \qquad s_{33} = s^*_{33}$$

† In obtaining the reduced constitutive equations (2.7) for an elastic material we assumed that the compliance tensor satisfied the condition $a_{ijkl} = a_{klij}$, which follows from the existence of an elastic strain energy function. In order to obtain viscoelastic equations of the same form for use in the correspondence principle, we must assume that $a_{ijkl}(t) = a_{klij}(t)$ for a viscoelastic material, which has no strain energy function. Experiments have been carried out which support this assumption for certain transversely isotropic materials, see [11].

in equation (6.6), we obtain

$$s_{33}^* = \frac{8I\hat{v}\,e^{-i\varepsilon}}{l^2M}. \tag{6.17}$$

Let $s_{33}^* = s_{33}' + is_{33}''$, where $s_{33}'(\omega)$ and $s_{33}''(\omega)$ are the real and imaginary parts of s_{33}^*, then it follows from (6.17) that

$$s_{33}' = \frac{8I\hat{v}\cos\varepsilon}{l^2M}, \qquad s_{33}'' = \frac{-8I\hat{v}\sin\varepsilon}{l^2M}.$$

Thus $s_{33}'(\omega)$, $s_{33}''(\omega)$ are obtained on measuring $\hat{v}$ and $\varepsilon(\omega)$ for a range of values of ω. We note that if $\varepsilon(\omega) = 0$, then $s_{33}'' = 0$ and s_{33}' is the elastic compliance of the material. A similar generalization of the other bending and torsion experiments makes it possible to determine all five complex compliances of a transversely isotropic viscoelastic material.

The beam bending stiffness and torsional rigidity

Returning to the elastic solution, we note that our analysis describes the deformation of an anisotropic elastic beam subjected to constant bending and twisting moments. In Section 5 we showed that a coupling exists between bending and twisting deformations of an anisotropic beam provided $a_{35} \neq 0$. Because of this coupling, we saw in Section 5(a) that the bending stiffness of a beam constrained not to twist ($\alpha = 0$) was greater than that of the same beam allowed to twist freely ($M_t = 0$), and in Section 5(b) there was a similar increase in torsional rigidity when bending was prevented. We now investigate these effects for an "off-axis" transversely isotropic beam where the magnitude of a_{35} (and hence the coupling) depends on the fibre angle θ.

Referring specifically to equations (5.14), (5.16) and (5.18), and by analogy with corresponding formulae for isotropic beams, we define

$$E_1 = \frac{1}{a_{33}}, \qquad E_2 = \frac{1}{a_{33}} + \frac{3a_{35}^2M^*}{4a_{33}^2}, \qquad G_1 = \frac{3M^*}{a_{33}(1+3a_{35}^2M^*/4a_{33}^2)}, \qquad G_2 = \frac{3M^*}{a_{33}}. \tag{6.18}$$

Then E_1 and E_2 are the effective longitudinal stiffnesses (or Young's moduli) for the beam in free bending ($M_t = 0$) and constrained bending where the beam may not twist ($\alpha = 0$). Similarly G_1 and G_2 are the effective shear moduli of the beam in free torsion with bending permitted and constrained torsion with bending prevented. Figure 3 shows the variation of E_1 and E_2 with fibre angle θ for a range of values of λ and for the composite material of (5.19). The lower curve E_1 is just the variation of longitudinal stiffness for the material with θ and is independent of λ. The remaining curves show the effective stiffness E_2 to be greater than E_1 as predicted. It is interesting to note that *for small λ and a range of values of θ, E_2 is actually greater than the maximum longitudinal stiffness of the beam material.* Similar results are shown in Fig. 4 where G_1 and G_2 are compared for two values of λ and a range of values of θ. We see that in each case $G_2 > G_1$. However, we cannot make quite the same physical interpretation in this case, since from equation (6.18) neither G_1 nor G_2 are identifiable as shear moduli for the elastic material, for they both depend on the geometrical configuration of the beam. We deduce from Fig. 4 only that the effective shear modulus of a particular beam is greater when the beam is so constrained that it cannot bend.

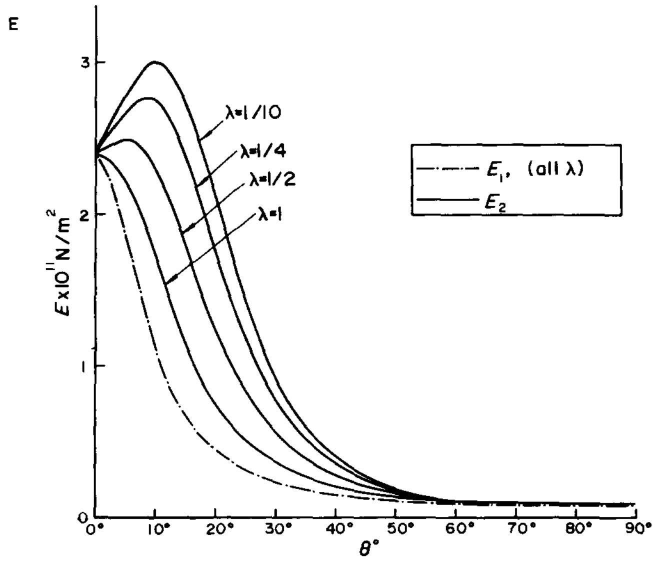

FIG. 3. Change in effective longitudinal stiffness for constrained bending.

These apparent changes in stiffness are not new types of phenomena; similar results exist in isotropic elasticity theory, but they have less practical significance. For example, if an isotropic elastic beam with Young's modulus E and Poisson's ratio σ is stretched longitudinally but so constrained as to prevent lateral contraction, the effective Young's

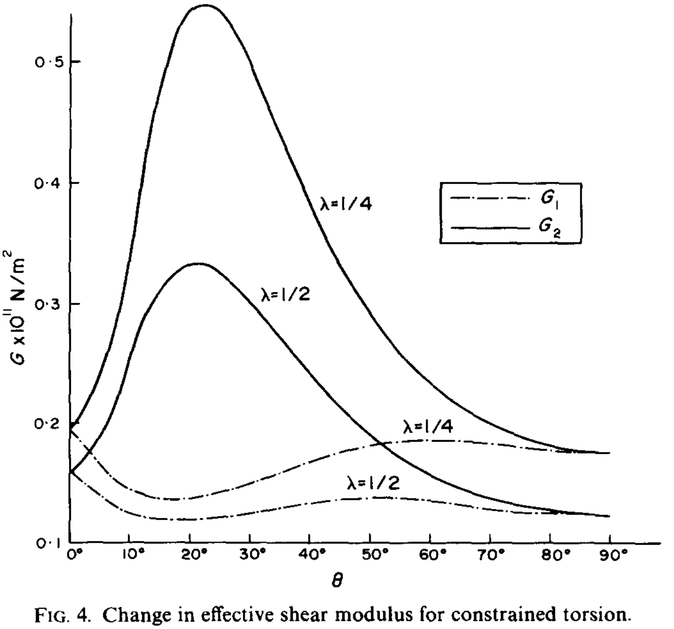

FIG. 4. Change in effective shear modulus for constrained torsion.

modulus has the value

$$E' = \frac{(1-\sigma)E}{(1-2\sigma)(1+\sigma)},$$

which is greater than E. However, this property cannot be utilized to increase the stiffness of a beam. The significance of our results is that it is possible to exploit the increased bending stiffness in practice since twisting may often be prevented.

Acknowledgements—The author is indebted to Dr. G. D. Dean, Mr. T. A. E. Gorley and Dr. F. J. Lockett for useful discussions and to Mr. G. F. Miller and Miss S. M. Picken for their assistance with the numerical work.

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(*Received* 3 *April* 1972; *revised* 24 *November* 1972)

Абстракт—Дается математическая формулировка изгиба и кручения анизотропной упругой балки. Путем применения вариационных методов, получаются приближенные аналитические решения, которые соглашаются хорошо с численным решением. Используются результаты для анализа изгиба и кручения поперечно изотропной, упругой балки. Указано, что жесткость изгиба увеличивается когда препятствовается укручение результат очень важный для расчета конструкций из составных материалов. Приводится последовательность экспериментов для изгиба и кручения, из которых можно определить пять податливостей для поперечно изотропного материала. Обобщаются результаты, с целью определения частоты зависимых податливостей для анизотропных, вязкоупругих материалов.